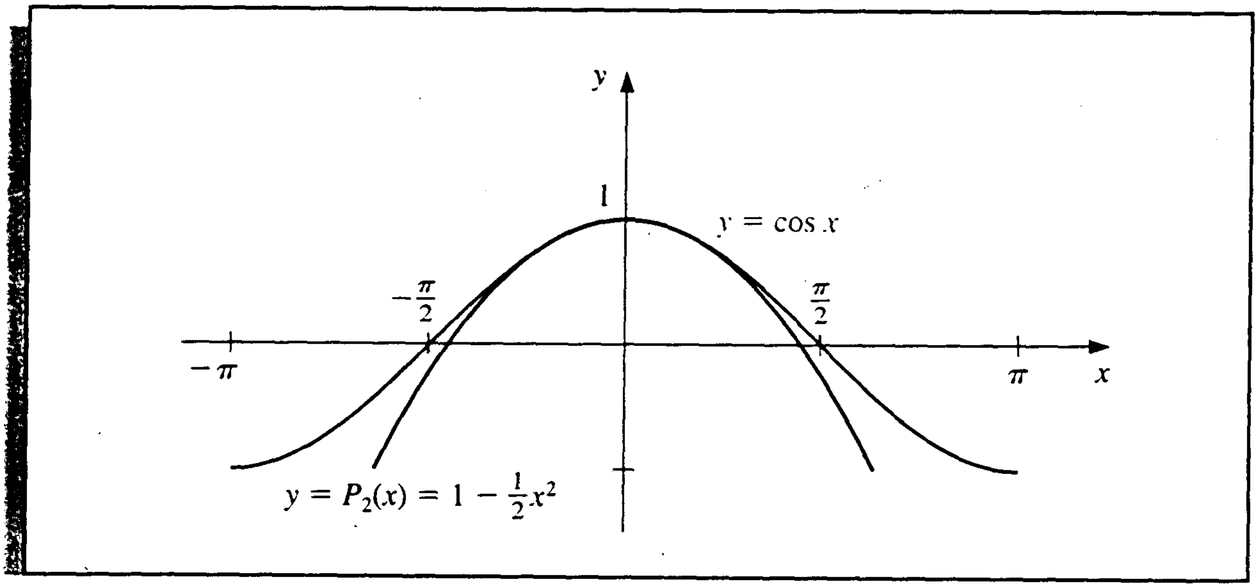
Since $f \in C^\infty(\mathbb{R})$, Taylor's Theorem can be applied for any $n > 0$. Also, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, and $f^{(4)}(x) = \cos x$, so $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$ and $f'''(0) = 0$.

a. For $n = 2$ and $x_0 = 0$, we have

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin \xi(x),$$

where $\xi(x)$ is a number between 0 and x. (See Figure 1.10.)

e 1.10



With $x = 0.01$, the Taylor polynomial and remainder term are

$$\begin{aligned}\cos 0.01 &= 1 - \frac{1}{2}(0.01)^2 + \frac{1}{6}(0.01)^3 \sin \xi(x)\\ &= 0.99995 + (0.1\bar{6}) \times 10^{-6} \sin \xi(x),\end{aligned}$$

where $0 < \xi(x) < 0.01$. (The bar over the 6 in $0.1\bar{6}$ is used to indicate that this digit repeats indefinitely.) Since $|\sin \xi(x)| < 1$ for all x, we have

$$|\cos 0.01 - 0.99995| \le 0.1\bar{6} \times 10^{-6},$$

so the approximation 0.99995 matches at least the first five digits of cos 0.01. Using standard tables we find that $\cos 0.01 = 0.99995000042$, so the approximation actually gives agreement through the first nine digits.

The error bound is much larger than the actual error. This is due in part to the poor bound we used for $|\sin \xi(x)|$. It can be shown that for all values of x, we have $|\sin x| \le |x|$. Since $0 < \xi < 0.01$, we could have used the fact that $|\sin \xi(x)| \le 0.01$ in the error formula, producing the bound $0.1\bar{6} \times 10^{-8}$.